Universality under conditions of self-tuning

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We study systems with a continuous phase transition that tune their parameters to maximize a quantity that diverges solely at a unique critical point. Varying the size of these systems with dynamically adjusting parameters, the same finite-size scaling is observed as in systems where all relevant parameters are fixed at their critical values. This scheme is studied using a self-tuning variant of the Ising model. It is contrasted with a scheme where systems approach criticality through a target value for the order parameter that vanishes with increasing system size. In the former scheme, the universal exponents are observed in naïve finite-size scaling studies, whereas in the latter they are not.

One historical motivation for the study of critical phenomena are the observable effects of diverging correlation lengths, such as critical opalescence [1]. On a more fundamental level, universality, the fact that a variety of phenomenologically different systems share the exact same critical behavior, reveals a deeply engrained mathematical structure in physical systems.

Universality is well understood in equilibrium systems where renormalization group methods can be applied. Far-from-equilibrium systems, often only described by dynamical rules, do not always lend themselves to the same methods of analysis, and as a consequence the understanding of universality is less complete. Some systems add another complication: they self-tune. We ask what happens to universality under conditions of self-tuning. In particular, we investigate in this paper a self-tuning mechanism that reproduces the universal finite size scaling of thermodynamic observables. While we investigate these issues in a near-equilibrium system, the arguments we put forward may well be applicable to the far-from-equilibrium systems typically studied in the literature on self-organized criticality (SOC).

The term self-organized criticality has been used with many different meanings in different disciplines, often simply to describe a system whose internal dynamics lead to a degree of scale freedom in global observables. A more specific definition of SOC, which we employ here, is the spontaneous emergence of critical behavior in systems with continuous phase transitions. For instance, sandpile models have been described in these terms [2–4]. In these systems, defined on d-dimensional lattices, local rules demand the toppling of particles to neighboring sites whenever a threshold value of the local particle density is exceeded. The boundaries are open such that particles can be dissipated, and a slow drive is implemented as an addition of a particle whenever the system reaches a globally stable state (no supercritical local particle densities). In these systems the distribution of avalanche sizes, defined as the number of local reconfigurations in response to the addition of a particle, is scale-free. Moments of the distribution show simple finite-size scaling [5], just like moments of the order-parameter distribution in equilibrium critical phenomena [6].

In 1988 Tang and Bak linked their sandpile model to ordinary non-equilibrium phase transitions [2]. The overall particle density, ζ, was identified as the tuning parameter and the density of active sites, ρ_a, also called the "activity", as the order parameter. Both are common observables in continuous phase transitions, and their identification enables the use of the powerful formalism of critical phenomena. Investigations of avalanche size distributions, which are characteristic of the smaller body of literature on SOC, have been developed less extensively. We use a notation inspired by absorbing-state (AS) phase transitions [4], for a review see [7] and references therein. Below a critical value, ζ_c, of the tuning parameter, the order parameter tends to zero since local thresholds are rarely surpassed anywhere in the system and hardly any topplings occur. The order parameter shows very good finite-size scaling, identical to that of corresponding AS phase transitions. In these corresponding models the boundaries are closed, and one measures quasi-stationary values of activities at fixed particle densities, ζ [8, 9]. In other words, there is ample numerical evidence supporting the fact that standard observables such as the order parameter in sandpile models respect the universality classes of their corresponding phase transitions. This is also reflected in the observation that avalanche-size exponents are directly related to the scaling exponents describing the order parameter, correlation length, and survival time distribution in the corresponding AS systems [10].

Sandpiles are defined in terms of their microscopic dynamics. It is desirable to mirror the effect of these dynamics in a general scheme that can be applied to any continuous phase transition. One of the most natural such schemes is to have the order parameter feed back on the tuning parameter [3, 7]. Indeed, such coupling can force an approach to the critical point as the lin-

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ear system size L diverges [11], and in sandpile models a narrative of such a coupling seems natural: Driving the system increases the tuning parameter until the critical point, ζ_c, is reached and activity ensues. Activity then leads to diffusion-like motion of particles through the system and to dissipation at the boundaries, that is, a reduction of the tuning parameter to below its critical value.

This general scheme can be summarized by the equation of motion [4]

$$\partial_t \zeta = h - \epsilon \rho_a(\zeta, t; L), \tag{1}$$

where h is a driving rate and ϵ represents a coarse-grained, or bulk, dissipation. The activity, ρ_a, depends on time t, and is treated as a noise term. The average order parameter $\langle \rho_a \rangle = \frac{h}{\epsilon}$ is readily obtained from the stationary state of this equation. As long as $\langle \rho_a \rangle$ vanishes in the thermodynamic limit, $L \to \infty$, criticality can be reached. However, the description of SOC as a result of a linear coupling between order and tuning parameter does not constrain the dynamics sufficiently to explain the identity of scaling behaviors. In this scheme (unlike in sandpile models), naïve finite-size scaling analyses of thermodynamic observables do not show the universal scaling exponents. By "naïve" we mean straight-forward numerical measurements of observables like $\langle \rho_a \rangle (L)$, as performed in SOC sandpiles, without taking into account (or knowledge of) the scaling of the driving and dissipation, see below. To naïvely observe universal finite-size scaling without having to account for the details of the tuning dynamics it would be necessary that $\langle \rho_a(L) \rangle \propto L^{-\beta/\nu}$, where β and ν are the exponents of the underlying phase transition. This can only be true if the system-size dependence of h and ϵ (introduced by the facts that the density of dissipating boundary sites vanishes as L^{-1} and avalanche durations increase with L such that $\lim_{L\to\infty} h(L) = 0$) is such that $h(L)/\epsilon(L) \propto L^{-\beta/\nu}$. In the coarse-grained description of coupled order and tuning parameters there is no reason why this last condition should be fulfilled. Assuming power laws $h(L) = h_0 L^{-\kappa}$ and $\epsilon(L) = \epsilon_0 L^{-\omega}$, we obtain scaling but not with the universal exponents (assuming any functional form other than power laws produces no scaling at all). There is an undetermined parameter, $\kappa - \omega$, which must be chosen by design to force the universal finite-size scaling exponents of the underlying phase transition upon the self-tuning system. Paradoxically, the scheme produces a system whose tuning parameter does assume its critical value in the thermodynamic limit, yet the exponents derived from naïve finite-size scaling analyses are non-universal. This problem was described in detail in ref. [11], and further discussed in ref. [12, 13]. Exponents estimated at criticality, *e.g.* from the spatial decay of correlation, or from critical scaling need not be affected by an arbitrary choice for $\kappa - \omega$.

In the present study we explore a scheme that achieves two goals: The tuning parameter reaches the critical point in the thermodynamic limit, and the exponents derived from naïve finite-size scaling analyses are identical to those of the underlying phase transition. Our goal is to understand the interplay between naïvely observed universality and the dynamics of self-tuning in generic systems. Here we choose a near-equilibrium system, but the findings may also be informative for far-from-equilibrium systems, like sandpile models.

The problem with coupling the tuning parameter to the order parameter can be understood as follows. The system adjusts its tuning parameter according to Eq. (1) to achieve, in the thermodynamic limit, the order parameter zero. For finite systems, however, a non-zero value is targeted, which is arbitrary to the extent that $h(L)/\epsilon(L)$, or equivalently $\kappa - \omega$, is arbitrary. The finite value is necessary because at the level of description of Eq. (1) fluctuations in $\rho_a(t; L)$ ensure that $\langle \rho_a \rangle (L)$ is always finite, even for $\zeta = 0$, for finite systems. Targeting zero (setting $h/\epsilon = 0$) would drive the system to zero tuning parameter and make SOC impossible. It may be argued that this constitutes a problem in the level of description and that in sandpile models the value $\rho_a = 0$ is reached in finite systems since the absorbing phase (unlike the high-temperature phase in an Ising model) has no fluctuations. Typically, however, the order parameter ρ_a, is defined as the asymptotic (long time average) value of the density of active sites, conditioned on the existence of active sites, see *e.g.* [8]. It is therefore always non-zero. In other words a "complete descripition" would require a re-definition of ρ_a and a revision of the entire formalism developed so far. Our approach is independent of the presence of fluctuations in the phase of vanishing order parameter.

Tuning the order parameter to zero as L diverges leaves an undesired arbitrariness in the intermediate values. Instead, we use the most prominent signals of criticality, *i.e.* the critical singularities (*e.g.* susceptibility, heat capacity, correlation length). The advantage of coupling the tuning parameter to such a singularity is that naïve finite-size scaling is universal, as numerical evidence suggests to be the case in SOC. This will be shown below.

For example we could couple the temperature T of a magnetic system to the susceptibility, $\chi(T; L)$ which diverges uniquely at the critical point, $\chi(T; L \to \infty) \propto |T \to T_c|^{-\gamma}$. This could be described by an equation of motion for the temperature

$$\partial_t T = k \partial_T \chi(T, t; L), \tag{2}$$

where k is a constant that is related to the relaxation time and needs to vanish in the thermodynamic limit to prevent destabilization. Under sufficiently slow dynamics, the stationary temperature of Eq. (2) will be that of maximum susceptibility, where $\partial_T \chi(T; L) = 0$. As is well known, the position in T of the peak of the susceptibility, $T_{\chi_{\max}}(L)$, approaches the critical point as $|T_{\chi_{\max}}(L) - T_c| \propto L^{-1/\nu}$. A derivation of this result from scaling arguments based on the renormalization group can be found in ref. [14], p. 72. The prefactor determining the shift in $T_{\chi_{\max}}(L)$ for finite L depends on the

boundary conditions of the system, but for our argument only the scaling behavior is important. If the system is well characterized by the average temperature it assumes under these dynamics, $\langle T \rangle$, it will be equivalent to a system approaching T_c with the exponent needed to naïvely observe the universal scaling behaviour, *i.e.* that of a system fixed at temperature $T = T_c$, while L diverges. In Monte Carlo studies of systems with unknown critical tuning parameter values, *e.g.* ref. [15], similar arguments ensure that scaling exponents can be derived numerically.

With $T(L) - T_c \propto L^{-1/\nu}$, we find $\langle |m| \rangle (T(L)) \propto (T(L) - T_c)^{\beta} \propto L^{-\beta/\nu}$, which is the universal scaling behavior. The same argument holds for all other thermodynamic observables, and equivalent results are obtained if we couple to any other quantity that diverges uniquely at T_c. Thus, by coupling the tuning parameter to the susceptibility the naïve finite-size scaling becomes fully universal.

The above ideas are implemented by modifying a 2-d Ising model and allowing it to adjust its dimensionless coupling constant, $K = J/(k_B T)$, where J is a ferromagnetic Ising coupling, and k_B is Boltzmann's constant. Physically, this corresponds to adjusting the temperature. We want to design a system that finds the maximum of the susceptibility with a precision that only depends on one scale, which may be linked to the system size. Instead of measuring the first derivative $\partial_T \chi$ required in Eq. (2) we choose the more robust method of bracketing to approach the maximum, see *e.g.* ref.[16], Chapter 9. To this end, three systems of equal size and shape but different initial temperatures $K_1 < K_2 < K_3$ are run simultaneously for $t_{\max}$ Monte Carlo steps. The first half of this time is used for equilibration; in the second half the first and second moments of the magnetization are recorded, and from them the susceptibilities $\chi(K_i)$ are calculated.

Ordering the temperatures by the corresponding susceptibilities, there are now $3! = 6$ possible scenarios. These can be grouped into two cases:

If the ordering of temperatures indicates that the maximum has not been bracketed, *i.e.* $K_{\chi_{\max}} > K_3$ or $K_{\chi_{\max}} < K_1$, then the search range is widened by reassigning the temperature furthest away from $T_{\chi_{\max}}$. For example, if $K_{\chi_{\max}} > K_3$, then K_1 is shifted to $K_1' = K_3 \sqrt{K_2 K_3}/K_1$. Other choices are possible, *e.g.* $(K_3)^2/K_1$; the only requirements are that the reassignment do not introduce any special scales (as would be the case *e.g.* for the choice $K_3 + \Delta K$ with a fixed ΔK) and that it widen the search range. The choice made here is convenient for its numerical stability.

If the maximum appears to be bracketed, *i.e.* $K_1 < K_{\chi_{\max}} < K_3$, the search range is narrowed by halving the distance in log-space between the middle-temperature and the temperature furthest away from it. For example, if K_3 is to be reassigned, it is shifted to $K_3' = \sqrt{K_2 K_3}$

Iterating this method the system (consisting of three Ising models) will shrink its search range until the accuracy with which χ is estimated during $t_{\max}$ Monte Carlo steps forces it to widen the search range. Due to fluctuations and the finiteness of $t_{\max}$, the measured $\chi(K_i)$ remain stochastic variables, and the three temperatures will fluctuate around the true $T_{\chi_{\max}}$ for any given system size as exemplified in Fig. 1. Importantly, however, the range of these fluctuations can be made arbitrarily small by increasing $t_{\max}$, see Fig. 2. As L increases, the most likely and average temperatures approach T_c, while the distributions become narrower. Holding $t_{\max}$ fixed, one observes the expected finite-size scaling of the temperature, *i.e.* $|\langle T \rangle - T_c| \propto L^{-1/\nu}$ and all other thermodynamic observables, *e.g.* $\langle |m| \rangle \propto L^{-\beta/\nu}$, up to a certain system size where the accuracy with which $T_{\chi_{\max}}$ is estimated becomes insufficient. Eventually the system becomes unstable as the assumption $t_{\max} \gg$ equilibration time becomes invalid. In order for naïve finite size scaling to be universal without a bound on L, we require that $t_{\max}$ increase sufficiently fast. The exact minimum speed at which $t_{\max}$ must diverge with L depends on the chosen dynamics and is given by the dynamical exponent z. We require

$$t_{\max} \propto L^x, \tag{3}$$

where $x \geq z$. This was only superficially confirmed by estimating the maximum equilibration times that destabilized systems of different sizes. In Fig. 3 numerical results of the most prominent observables are presented for system sizes up to $L = 512$, where $x = 2.15$ was used.

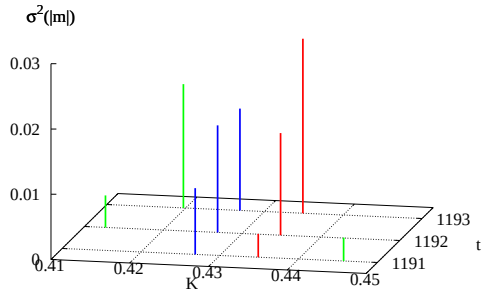

FIG. 1: Self-organizing couplings in a system of size $L = 64$. Each time step consists of an equilibration of 388 scans over the lattice and another 388 scans during which the susceptibility is determined. At $t = 1191$, the maximum is not bracketed and the search range is widened. In the next time step it is bracketed, and the range shrinks. Initially three temperatures far from criticality ($K_c \approx 0.44068$) are chosen that bracket the maximum, $K_1 = 0.3, K_2 = 0.5, K_3 = 0.6$. The systems converge near the temperature of maximum susceptibility. Statistics are collected after 1000 equilibration time steps.

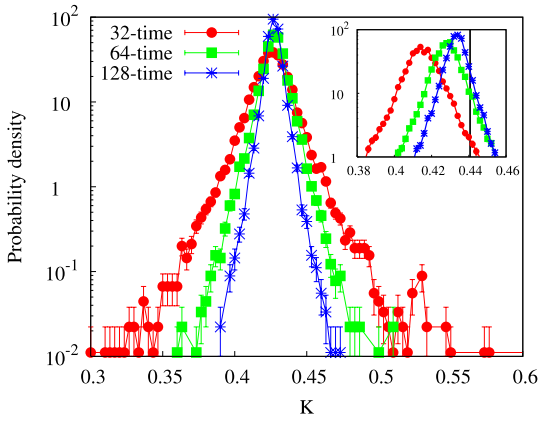

FIG. 2: The distribution of self-organized temperatures, can be made arbitrarily narrow by increasing $t_{\max}$, see main graph, where $t_{\max} = 0.4 \times 32^{2.15}, 0.4 \times 64^{2.15}, 0.4 \times 128^{2.15}$, for fixed $L = 32$. Inset: Temperature distributions for $L = 16, 32, 64$, where K is adjusted after a fixed $t_{\max} = 0.1 \times 128^{2.15}$.

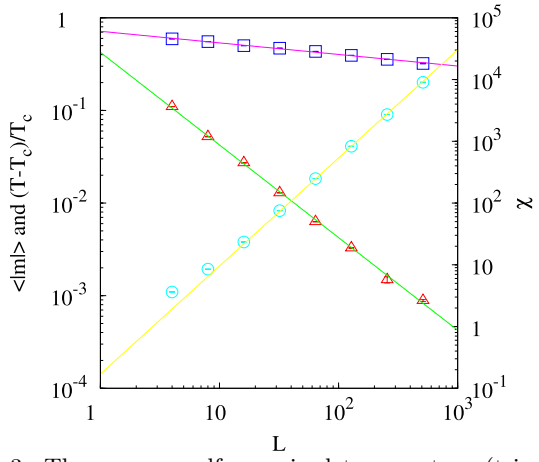

FIG. 3: The average self-organized temperature (triangles), magnetization (squares), and susceptibility (circles) in systems ranging from $L = 4, 8, 16...512$. Straight lines represent power laws with the known exponents of the 2-d Ising universality class, symbols are measurements.

For the single-spin flip dynamics used here, $z = 2.13(3)$ [17]. Stricly speaking, the macroscopic time referred to in Fig. 1 to estimate the position of the peak of the temperature distribution also has to increase with L. In the present study we are always far from the limit where this issue becomes important. Our situation is fundamentally different from that of the order parameter coupled to the tuning parameter, where there was only one special choice for the exponent $\kappa - \omega$ that reproduced the known finite-size scaling exponents. The new scheme is more robust since the exponent $x \geq z$ is only restricted to a semi-infinite range. This situation is very similar to sandpiles in the following sense: For a sandpile to display proper scaling for all system sizes, the intervals (measured in a microscopic time scale of individual topplings) between additions of grains must diverge with system size fast enough. However, it is impossible to drive a sandpile "too slowly". If the drive is slower than necessary, then the sandpile will be inactive for a while between avalanches, but the avalanches and associated order-parameter properties will still obey the universal scaling laws.

The idea for the present study emerged from a discussion of biological evolutionary systems that are believed to maximize a form of susceptibility, as they balance the need for rigidity to store information (for example in the form of DNA) against the need for flexibility to respond to new situations. While the use of multiple copies of a system may seem curious in the context of sandpiles, it is unavoidable in the context of evolution.

The correlation length, ξ, which also diverges at criticality, seems the most natural means to discuss the relevance of the proposed mechanism to sandpile models. It has been speculated that the correlation length plays an important role in the feedback between order- and tuning parameter [3]. The correlation length measures the spatial distance over which perturbations to a system can be communicated. In thermal systems ξ needs to be inferred from the spatial decay of correlation functions. In sandpiles, on the other hand, the length over which perturbations are communicated is dictated directly by the dynamics. Any particle added to a sandpile must be transported to the boundaries to be dissipated. The assumption of stationarity thus implies that perturbations can be felt across the entire system, *i.e.* $\xi \propto L$. Clearly, this corresponds to the maximization of ξ for any L. Imposing this relation, just like maximizing the susceptibility, leads to the preservation of naïvely observed universality, see also [11]. Therefore, thinking of sandpile models as correlation-maximizers rather than activity-minimizers, observations of the universal scaling exponents become the expectation rather than the surprise.

In conclusion, a mechanism has been investigated which does more than bring a system to the critical point. It reproduces the scaling exponents observed in finite-size scaling studies where a corresponding model is fixed at criticality. Thus the mechanism preserves universality in naïvely performed finite-size scaling analyses under conditions of self-tuning. This was achieved by letting the tuning parameter maximize a quantity that diverges uniquely at criticality. In contrast to a coupling between order and tuning parameter, this allows us to use the well-defined maximum rather than an arbitrary small parameter for orientation. The Ising model was used to show that the process is indeed capable of recovering both the well-known scaling exponents and the critical temperature.

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